Comparative Study of Newton-Raphson, Muller and Chebyshev Methods for Solving Polynomial Equation

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Abstract

This paper is based on the relative study of three iterative methods named Newton-Raphson, Muller and Chebyshev methods, the rate of convergence of every method will be analyzed after solving numerical problem by implementing each method independently.

Key words: *Newton-Raphson Method, Muller Method, Chebyshev Method, numerical example*

I. Introduction

In mathematics we deals with many polynomial equations of the form $p(x) = a_0x^r + a_1x^{r-1} + \cdots + a_{r-1} + a_r$, where a's are constants, $a_0 \neq 0$. If $p(x)$ constant some other function such that as trigonometric, logarithmic, exponential etc, then$p(x) = 0$ is a transcendental equation. If p be a continuous function. Any number ξ for $p(\xi) = 0$is a root of equation $p(x) = 0$.where,root of $p(x)$ is ξ. A root of ξ is called of multiplicity q, if $p(x) = (x - \xi)^q g(x)$, $g(x)$ is bounded at ξ and $g(\xi) \neq 0$. If $q = 1$, then ξ is said to be simple zero and if $q > 1$, then ξ is called a multiple zero.

Out of all the studied methods [1] has considered Secant method as the most effective one and has the converging rate near to N-R method but here only unique function evaluation required for each iteration and analyzed the slow rate of order of convergence of bisection method .Robin et al [2] explains the rate of convergence of Secant, Regula-Falsi, Newton-Raphson, Muller, Chebyshev methods respectively.Secant, Regula-Falsi and Newton-Raphson methods, which are based on 1st degree equations, Newton-Raphson has a good rate ofconvergence i.e. 2. Among Muller and Chebyshev methods,for solving 2nd degree equations, Chebyshev consideredbest using the rate of convergence of 3. Abdulaziz Ahmad [3] compared Newton Rapson method with Bisection method in term of it order of convergence. In Bisection method the value of the root is grouped within the bound of interval, so the method is assured to converge but is very slow.This is results shows that its converging rate is close to Newton-Rhapson method, but necessitates only a single function evaluation per iteration. Furthermore it can be concluded that though the convergence of Bisection is guaranteed,but with a slow convergence rate and as such it is quite hard to extend to use for systemof equations compared to Newton's Rhapson method Azizul et al [4] analyzedthe most effective of the new method in its study. It only requires a single function evaluation per iteration Analysis of efficiency from the numerical computation shows that this method are preferable to the well-known Newton's method.

Order of Convergence

consider$\{b_n\}_{n=0}^{\infty}$ is a sequence which is converges to b, with $b_n \neq b$for every n. If positive constants λ and β exist with

$$\lim_{n\to\infty} \frac{|b_{n+1} - b|}{|b_n - b|^{\beta}} = \lambda \,,$$

then $\{b_n\}_{n=0}^{\infty}$ converges to b of order β,with asymptotic error constant λ. The order of iterative technique of the form $b_n = g(b_{n-1)}$ is β, if the sequence $\{b_n\}_{n=0}^{\infty}$ converges to the solution $b = g(b)$ of order β. Usually, higher order of convergence sequence converges more swiftly than with a lower order sequence. The speed of convergence affected by the asymptotic constant but degree of the order remains unaffected. If $\beta = 1$ (and $\lambda < 1$), the sequence is linearly convergent. And is called quadratically convergent for $\beta = 2$.The different numerical methods have different rate of convergence, the rate of convergence is a very important topic in solution of polynomial and transcendental equations, because the rate of convergence of any numerical method determines the speed of the approximation to the solution of the problem. Thus the order of convergence of Newton-Raphson method is 2 that are Newton-Raphson is quadric convergent, convergence order of Muller method is 1.84 and that of Chebyshev3.

II Discussion of Methods:

1.Newton-Raphson

. When the derivative of $f(y)$ can be Easley found, the real root of the $f(y) = 0$ can be compute quickly by Newton-Raphson method. Let y_0 be approximate root of $f(y) = 0$ and let $y_1 = y_0 + h$ be the correct root where$f(y_1) = 0$. where$f(y_0 + h) = 0$ by Taylor's series, $f(y_0) + \frac{h}{1!}\acute{f}(y_0) + \frac{h^2}{2!}f''(y_0) + \cdots = 0$, since h is small, h^2 and higher powers of h may be omitted. Hence $f(y_0) + h\,\acute{f}(y_0) = 0$ a better approximations given by$y_2, y_3, \dots, y_{j+1}$, where

$$y_{j+1} = y_j - \frac{f(y_j)}{\acute{f}(y_j)} \quad , j = 0,1,2, \dots \tag{1}$$

It is called Newton-Raphson formula.

2. Muller method

Muller method is an iterative method in which do not require derivative of the function.Muller method is useful in finding the roots of polynomials. It is a similarity of the secant method. In this method the function $f(x)$ is approximated of the root. This method is as fast as newton method and can be used to find real or complex zeros of a function. Let x_{j-2}, x_{j-1} and x_j be the approximation to a root of the equation $f(x) = 0$ and let$y_{j-2} = f(x_{j-2})$,$y_{j-1} = f(x_{j-1})$ and $y_j = f(x_j)$, then

$$y = A(x - x_j)^2 + B(x - x_j) + y_j \tag{2}$$

Be (2) the parabola passing through $(x_{j-2}, y_{j-2}), (x_{j-1}, y_{j-1})$ and (x_j, y_j).therefore we have

$$y_{j-1} = A(x_{j-1} - x_j)^2 + B(x_{j-1} - x_j) + y_j$$

$$y_{j-2} = A(x_{j-2} - x_j)^2 + B(x_{j-2} - x_j) + y_j$$

And so,

$$y_{j-1} - y_j = A(x_{j-1} - x_j)^2 + B(x_{j-1} - x_j) \tag{3}$$

$$y_{j-2} - y_j = A\left(x_{j-2} - x_j\right)^2 + B\left(x_{j-2} - x_j\right) \tag{4}$$

Solving (3) and (4) for *A* and *B*, we get

$$A = \frac{\left(x_{j-1} - x_j\right)\left(y_{j-1} - y_j\right) - (x_{j-1} - x_j)(y_{j-2} - y_j)}{(x_{j-1} - x_{j-2})(x_{j-1} - x_i)(x_{j-2} - x_j)} \tag{5}$$

$$B = \frac{\left(x_{j-2} - x_j\right)^2\left(y_{j-1} - y_j\right) - \left(x_{j-1} - x_j\right)^2(y_{j-2} - y_j)}{(x_{j-2} - x_{j-1})(x_{j-1} - x_j)(x_{j-2} - x_j)} \tag{6}$$

The quadric equation $A(x_{j-1} - x_j)^2 + B\left(x_{j-1} - x_j\right) + y_j = 0$, with A and B given by (5) and (6) yields the next approximationx_{j+1}, as

$$x_{j+1} - x_j = \frac{-B \pm \sqrt{B^2 - 4Ay_j}}{2A}$$

$$x_{j+1} = x_j - \frac{2y_j}{B \pm \sqrt{B^2 - 4Ay_j}} \tag{7}$$

The sign in the denominator is selected so that the denominator becomes largest in magnitude.

3. Chebyshev method

The Chebyshev method is in iterative method for determining the solution of a system of linear equations. Let for function $f(x)$ a polynomial of degree two in the form

$$f(x) = a_0x^2 + a_1x + a_2 = 0\,, \qquad a_0 \neq 0 \tag{8}$$

Determine a_0, a_1 and a_2 in (1), using condition

$$f_j = a_0x_j^2 + a_1x_j + a_2$$

$$\acute{f}_j = 2a_0x_j + a_1 \tag{9}$$

$$f_j'' = 2a_0$$

From (8) and (9) on eliminating $a_i's$, we get

$$f_j + \left(x - x_j\right)\acute{f}_j + \frac{1}{2}(x - x_j)^2 f_j'' = 0\,, \tag{10}$$

This is the Taylor series extension of $f(x)$ about $x = x_j$ such that the terms of order $(x - x_j)^3$ and greater powers are neglected. We can solve easly (10) is a quadric equation. In order to get the next approximation to the correct root from (10) we have

$$x_{j+1} - x_j = -\frac{f_j}{\acute{f}_j} - \frac{1}{2}(x_{j+1} - x_j)^2 \frac{f_j''}{\acute{f}_j} \tag{11}$$

For $(x_{j+1} - x_j)$ for $x_{j+1} = x_j - \frac{f_j}{\acute{f}_j}$ by $(-\frac{f_j}{\acute{f}_j})$on the right side of (11) , we get

$$x_{j+1} = x_j - \frac{f_j}{\acute{f}_j} - \frac{1}{2}\frac{f_j^{\,2}}{f_j''^{3}} f_j'' \tag{12}$$

It is the iterative formula of Chebyshev method.

III. Numerical Experiments

The polynomial equations can be considered.

Problem 1

We can Solve the$y^3 - 3y - 5 = 0$apply Newton-Raphson method.

We find the root of $f(y) = y^3 - 3y - 5$. We have$\acute{f}(y) = 3y^2 - 3$ .Since$f(0) = -5$, $f(1) = -7$, $f(2) = -3$ and $f(3) = 13$, we know that root lies between 2 and 3. We try a starting value of $y_0 = 3$, since $f(y_0) = 13$ and $\acute{f}(y_0) = 24$,from (1), we obtain

Table 1. Iteration data for Newton-Raphson method

j	y_j	$f(y_j)$	$\acute{f}(y_j)$	y_{j+1}
0	3	13	24	2.45833
1	2.45833	2.48165	15.13016	2.29431
2	2.29431	0.19399	12.79158	2.79158
3	2.27914	0.00153	12.58344	2.27902
4	2.27902	0.00002	12.5818	2.27902

Problem 2

We can Solve the equation $x^3 - 3x - 5 = 0$ apply Muller method.

We note that $f(0) = -5$, $f(1) = -7$, $f(2) = -3$ and $f(3) = 13$, we know that root lies between 2 and 3, we have $x_{j-2} = 1$, $x_{j-1} = 2$, $x_j = 3$, $y_{j-2} = -7$, $y_{j-1} = -3$and $y_j = 13$, from equations (5) and (6), we have

Table 2. Iteration data for Muller method

i	(x_j)	y_j	A	B	x_{j+1}
0	3	13	6	22	2.25957
1	2.25957	-0.24212	7.25969	12.50939	2.27871
2	2.27871	-0.00388	7.53914	12.59393	2.27902
3	2.27902	0.00002	0/0	0/0	2.27902

Problem 3

We Solve the equation$x^3 - 3x - 5 = 0$, apply Chebyshev, method.

We finding the root of$f(x) = x^3 - 3x - 5$. We note that $\acute{f}(x) = 3x^2 - 3$ and$f''(x) = 6x$. We try a starting value of $x_0 = 3$, since $f(x_0) = 13$, $\acute{f}(x_0) = 24$and $f''(x_0) = 18$, from (12), we obtain

Table 3. Iteration data for Chebyshev method

j	(x_j)	$f(x_j)$	$\acute{f}(x_j)$	$f''(x_j)$	x_{j+1}
0	3	13	24	18	2.34830
1	2.34830	0.90483	13.54354	14.0898	2.27917
2	2.27917	0.0019	12.58385	13.67502	2.27902
3	2.27902	0.00002	12.5818	13.67941	2.27902

IV Conclusion:

The rate of convergence is a very important topic in solution of polynomial and transcendental equations, for any numerical method determines the speed of the approximation to the solution of the problem.The Newton-Raphson method hassecond order quadratic convergence. Muller method of iteration converges quadraticallyalmost for all initial approximations.We can be used to compute complex roots.In Muller method,if no better approximations are known, we can put$x_{j-2} = -1, x_{j-1} = 0$ and $x_j = 1$.The most commonly used method is Newton-Raphson method,once the initial value of the root have been found near the actual root, the convergence of this method is faster. If the choice of the initial value of the root is poor, the actual root may not be obtained. This method and Chebyshev method can be applied only when $\acute{f}(x)$ and$f''(x)$ can be found out. If $\acute{f}(x)$ is near zero for the approximate root of $f(x) = 0$, then these methods are not suitable.Newton-Raphson method can be extendedto systems of nonlinear equations. Here again probability of success is high, only if good starting solution is assumed.On comparing above three methods, we conclude that Chebyshev and Muller methods have less number of iteration, so these are more efficient. Further, Chebyshev has order of convergence 3, which is greatest of all three. Hence Chebyshev is most effective out of these three methods.

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